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Twist operator correlation functions in $O(n)$ loop models

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Abstract

Using conformal field theoretic methods we calculate correlation functions of geometric observables in the loop representation of the $O(n)$ model at the critical point. We focus on correlation functions containing twist operators, combining these with anchored loops, boundaries with SLE processes and with double SLE processes. We focus further upon $n = 0$, representing self-avoiding loops, which corresponds to a logarithmic conformal field theory (LCFT) with $c = 0$. In this limit the twist operator plays the role of a 0-weight indicator operator, which we verify by comparison with known examples. Using the additional conditions imposed by the twist operator null states, we derive a new explicit result for the probabilities that an $SLE_{8/3}$ winds in various ways about two points in the upper half-plane, e.g. that the SLE passes to the left of both points. The collection of $c = 0$ logarithmic CFT operators that we use deriving the winding probabilities is novel, highlighting a potential incompatibility caused by the presence of two distinct logarithmic partners to the stress tensor within the theory. We argue that both partners do appear in the theory, one in the bulk and one on the boundary and that the incompatibility is resolved by restrictive bulk–boundary fusion rules.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It is well established that many two-dimensional statistical physics systems, including the $O(n)$ and percolation models which we consider in this paper, can be mapped to an equivalent loop representation [1]. Much of the recent interest in these loop representations of statistical systems has been motivated by the success of Schramm–Loewner evolution (SLE) in describing

loops emanating from boundaries [2, 3]. Progress has also been made in quantifying the effects of the fluctuations due to background loops, dubbed the ‘loop-soup’, via conformal loop ensembles (CLE) [4].

Alternately the loop model can be mapped to a height model via the Coulomb gas, which renormalizes onto a conformal field theory (CFT) when taken to the continuum limit. This allows us to study these loop models using the powerful methods of CFT [5, 6]. However, the unitary minimal model CFTs that are most familiar from the study of statistical mechanics are insufficient to describe the full behavior that can be observed in the loop models. We are forced to consider the extended Kac table, which implies the existence of logarithmic modules in the CFT. Logarithmic (L)CFTs have been studied extensively [7, 8], but little progress has been made in completing the association to general loop models.

In [9] operators were identified in the Coulomb gas that when applied in sets of two, change the weight of all loops that separate the insertion points. When the new weight is chosen to be the negative of the original we have the twist operators, first studied in [10]. Inserting a pair of twist operators modifies the partition function with a positive or negative weight for each configuration depending on the parity of the number of loops crossing a defect line between the two points. When background loops are suppressed (such as in the $n = 0$ limit of the $O(n)$ model, which is identified with self-avoiding walks and loops and SLE/CLE parameter $\kappa = 8/3$) the correlation functions can be used either directly or through a small n expansion to determine the topological properties of loops with respect to the twist points.

Regularization issues arise associated with the scaling properties of small loops in the neighborhood of the twist operators. In [10] these issues are addressed by inserting additional twist operators and isolating those solutions where loops separate pairs of these operators. The distance between the pairs of operators sets a scale preventing the profusion of small loops.

In this paper we fix the scale by other means: in the bulk we fix a scale for our loops using 2-leg operators that ensure that the loops pass through two given points. The result is equation (49) the probability within the ensemble of self-avoiding loops anchored at z_3 and z_4 that the loop separates z_1 from z_2 .

In regions with boundaries we use boundary N -leg operators to set our scale. We begin with a pair of 1-leg or SLE operators, and include only one twist operator, which marks parity against the boundary, and recover Schramm’s result for the left crossing SLE probability [11] at $\kappa = 8/3$. Next we include a result for two self-avoiding walks anchored at common points using a pair of boundary 2-leg operators along with a twist operator. We see that at $\kappa = 8/3$ this is equivalent to a result first reported in [12] for double SLE.

We then extend Schramm’s result for the self-avoiding walk by including a second twist operator which allows us to determine the weights of $SLE_{8/3}$ conditioned to wind about two points in any of the four possible ways. For example, given an $SLE_{8/3}$ process in the upper half-plane from 0 to infinity the probability of a *double* left passage with respect to the two points $r_A \exp(iv_A)$ and $r_B \exp(iv_B)$ is

$$\cos^2(v_A/2) \cos^2(v_B/2) + \frac{\sin(v_A) \sin(v_B)}{4} \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right], \tag{1}$$

where we have defined

$$\sigma = \frac{r_A^2 - 2r_A r_B \cos(v_A - v_B) + r_B^2}{r_A^2 - 2r_A r_B \cos(v_A + v_B) + r_B^2}. \tag{2}$$

This result is the half-plane version of (77); equations (77)–(80) convey all four of the winding probabilities in either the upper half-plane or infinite strip coordinates.

This analysis also provides an additional example of a physically meaningful $c = 0$ LCFT. Recently a great deal of attention has been paid to the boundary CFT of critical percolation

[13–15], which is an example of a $c = 0$ LCFT generated by a zero weight module with a second-order null state. This CFT possesses a transparent interpretation in terms of the equivalent SLE_6 arc representations. In the $O(n = 0)$ model the chiral components of a twist operator belong to the same LCFT module as the percolation SLE operator. Thus the $O(n = 0)$ twist operators give us a second physical picture with which to probe this LCFT.

Because the operators we consider are stationary in the Kac table as we continuously vary the loop fugacity of the model, we expect that our correlation functions behave qualitatively like analogous functions in $c > 0$ theories, obeying null state differential equations and exhibiting factorization between holomorphic and anti-holomorphic sectors. In spite of an incomplete catalog of the $c = 0$ CFT operator content the assumption seems justified and we observe a simple way of combining the holomorphic and anti-holomorphic sectors in the bulk theory. We observe that the leading terms of fusions associated with bulk logarithmic modules tend to have nonzero spin. We identify one such fusion that yields a weight one, spin one, leading fusion product for all values of n , indicating the presence of a non-Kac zero weight staggered logarithmic module in all corresponding CFTs; this is precisely the type of module used in deriving Schramm’s formula [11].

In [14, 16] the point was made that at $c = 0$ the logarithmic theories generated by the two modules with second order null states, $\mathfrak{M}_{1,2}$ and $\mathfrak{M}_{2,1}$, are incompatible in the sense that the two point function between logarithmic operators from the two theories cannot be defined. This means that the two modules cannot both be present in the same chiral theory. Our calculation of (1) involves a correlation function containing bulk twist operators, $\phi_{2,1}$, and boundary SLE operators, $\phi_{1,2}$. The solutions that we find illustrate that the two modules can indeed coexist in this mixed case, with one appearing in the bulk sector and one in the boundary sector, so long as the bulk–boundary fusion products belong exclusively to the minimal subsector common to both modules.

In section 2 we calculate several twist operator correlation functions in the $O(n)$ model for general n . In section 3 we specialize these results to the self-avoiding loop ensemble. In section 4 we derive a new $SLE_{8/3}$ result by solving a six point chiral correlation function. In section 5 we discuss the implications of this chiral correlation function for the $c = 0$ LCFT. Our conclusions are in section 6.

2. The $O(n)$ loops with twist operators

We begin this section with a brief description of the loop representation of the $O(n)$ model and a quick summary of the steps required to associate it with a meaningful CFT. For a more complete treatment we refer the interested reader to the pertinent references.

The standard loop representation of the $O(n)$ model [1] begins with n -component spins $s(r_i)$ with squared norm n on each site of a lattice Ω governed by the partition function

$$Z_\Omega = \text{Tr} \prod_{\langle ij \rangle} (1 + x s(r_i) \cdot s(r_j)). \tag{3}$$

We expand the product and associate a graph on Ω to each term by including the bond between r_i and r_j if the $x s(r_i) \cdot s(r_j)$ factor appears in the factor and excluding the bond if it does not. Now the trace per site of an odd number of spin components is zero, so the only graphs that contribute are those composed entirely of closed loops.

The form of the resulting loop partition function is particularly simple if we choose the honeycomb lattice, HC , for Ω . In this case the loops can visit each site a maximum of one

time and the per site trace, $\text{Tr}_{s_a(r_i)s_b(r_i)} = \delta_{ab}$, means that each loop earns a net weight n . Combining this with the weight x per occupied bond yields the partition function

$$Z_{HC} = \sum_{\Lambda} n^{\mathcal{N}} x^L, \tag{4}$$

where the sum is over all loop configurations Λ on the honeycomb lattice, \mathcal{N} the number of loops and L the total length of the loops in each configuration. For small values of x , long loops are repressed and the model flows to the vacuum under renormalization. For large values of x long loops carry more weight and the system flows to a fixed point of densely packed loops under renormalization. The boundary between these two regimes is $x_c = (2 + \sqrt{2 - n})^{-1/2}$, for which the system is critical and flows to the dilute fixed point [1].

We can map the loop model to the Coulomb gas by replacing the sum over configurations, with a sum over configurations of *directed* loops with a complex local weight that recreates the factors related to the non-local observable \mathcal{N} . To recreate these factors we associate a weight $\exp(i\chi\theta/2)$ to each vertex containing a loop, where θ is the angle the loop turns though while traversing the vertex. Each loop must close on itself thereby picking up a total weight of $\exp(\pm i\chi\pi)$ depending on whether it is directed clockwise or counter-clockwise. Taking the trace over the directions in each directed loop configuration is equivalent to an undirected loop configuration with weight $2 \cos(\chi\pi)$ per loop, so that by careful selection of χ we recover the $O(n)$ model weights.

The directed loops are equivalent to level lines of a height variable on the dual of our lattice if we insist that the height variable increases (decreases) by π whenever we cross a loop pointing to the left (right). The power of the Coulomb gas formalism lies in the assumption that this height model flows into a Gaussian free field under renormalization, which allows us to make precise calculations in the continuum limit of these models using field theory techniques.

There has been a great deal of success in using operators in the height model to glean information about the associated loop model and to derive relations between the model and the CFT describing its continuum limit. We emphasize that our description of the mappings between these models is by no means complete, as there are a variety of subtleties that we either simplify or omit completely.

It can be argued that the two non-trivial fixed points of the $O(n)$ loop model correspond to SLE/CLE $_{\kappa}$ s with

$$n = 2 \cos\left(\frac{(\kappa - 4)\pi}{\kappa}\right), \quad \begin{cases} (2 < \kappa < 4) & \text{dilute} \\ (4 < \kappa) & \text{dense.} \end{cases} \tag{5}$$

Alternately the correspondence with the Coulomb gas implies that these loop models are described by CFTs with central charge and conformal weights given by

$$c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa} \quad \text{and} \quad h_{r,s} = \frac{(\kappa r - 4s)^2 - (\kappa - 4)^2}{16\kappa}. \tag{6}$$

In this paper we focus primarily on the dilute regime with $2 < \kappa < 4$ relating to the critical $O(n)$ model. In [10] twist operators were identified that change the weight of all loops that separate two twist operators so that the new weight associated with these loops is $-n$. The partition function becomes

$$Z = \sum_{\Lambda} (-1)^{N_s} n^{\mathcal{N}} x^L, \tag{7}$$

where N_s is the number of loops separating the twist operators. In [10] it was shown that the twist operator was a Kac operator with weight

$$h_{\text{twist}} = \bar{h}_{\text{twist}} = h_{2,1} = \frac{3\kappa - 8}{16}. \tag{8}$$

In this paper we utilize the second-order null-state descendant of this Kac operator to derive differential equations in twist operator correlation functions.

2.1. Twist operators and anchored $O(n)$ loops in the bulk

In order to avoid the complications that small loops cause to twist operators, we are interested in establishing a scale by anchoring our bulk loops with 2-leg operators. In [17], the exponent for these operators are calculated in the Coulomb gas formulation of the loop model. In the Coulomb gas the 2-leg operators are equivalent to the insertion of a vortex and anti-vortex, where the (anti-)vortex has two directed lines flowing into (out of) the point. Because of this, no path can have both of its ends connected to a single 2-leg operator and furthermore, the paths associated with these vortices have weight 1 not n . Using the Coulomb gas vertex association it can be shown [17, 18] that these operators have weight

$$h_{2\text{-leg}} = \bar{h}_{2\text{-leg}} = h_{0,1} = \frac{8 - \kappa}{16}, \tag{9}$$

where we use the Kac weight convention to give the weight of these operators even though the indices are not positive integers. This means that the 2-leg operators do not have null state descendants, nor related differential equations, for general κ .

The correlation function we calculate includes one pair of 2-leg operators and one pair of twist operators. As per the usual CFT approach, we focus on the holomorphic sector and later sew this together with its anti-holomorphic counterpart [5]. Conformal symmetry fixes the form of the correlation function as

$$\langle \phi_{2,1}(z_1)\phi_{2,1}(z_2)\phi_{0,1}(z_3)\phi_{0,1}(z_4) \rangle = z_{21}^{-2h_{2,1}} z_{43}^{-2h_{0,1}} F\left(\frac{z_{21}z_{43}}{z_{31}z_{22}}\right), \tag{10}$$

where we have used the shorthand $z_{ij} := z_i - z_j$.

The null state $(2(1 + 2h_{2,1})L_{-2} - 3L_{-1}^2)|\phi_{1,2}\rangle$ implies that the differential operator

$$\left(\frac{h_{0,1}}{z_{41}^2} - \frac{\partial_{z_4}}{z_{41}} + \frac{h_{0,1}}{z_{31}^2} - \frac{\partial_{z_3}}{z_{31}} + \frac{h_{2,1}}{z_{21}^2} - \frac{\partial_{z_2}}{z_{21}}\right) - \frac{3\partial_{z_1}^2}{2(1 + 2h_{2,1})}, \tag{11}$$

annihilates the correlation function [5]. If we apply this differential operator to the correlation function and take the standard limit $\{z_1, z_2, z_3, z_4\} \rightarrow \{0, x, 1, \infty\}$ then we recover the following condition for $F(x)$

$$0 = F''(x) + \frac{(8 - 2\kappa) - (8 - \kappa)x}{4x(1 - x)} F'(x) - \frac{\kappa(8 - \kappa)}{64(1 - x)^2} F(x). \tag{12}$$

Solving (12) we find the $x \approx 0$ conformal blocks

$$F_{1,1}(x) = (1 - x)^{-\kappa/8} {}_2F_1\left(-\frac{\kappa}{4}, \frac{4 - \kappa}{4}; \frac{4 - \kappa}{2}; x\right) \quad \text{and} \tag{13}$$

$$F_{3,1}(x) = x^{\kappa/2-1} (1 - x)^{-\kappa/8} {}_2F_1\left(\frac{\kappa - 4}{4}, \frac{\kappa}{4}; \frac{\kappa}{2}; x\right), \tag{14}$$

where the subscript indicates the related fusion product as $z_2 \rightarrow z_1$. The fusion products in $\phi_{2,1} \times \phi_{2,1} = \mathbf{1} + \phi_{3,1}$ are the identity and the leading order energy density operator.

In contrast, there is an ambiguity in the conformal blocks as $z_2 \rightarrow z_3$. The fusion products in $\phi_{2,1} \times \phi_{0,1} = \phi_{1,1} + \phi_{-1,1}$ are operators with weights 0 and 1 for all values of kappa. The weights of these operators differ by an integer and the conformal blocks exhibit logarithmic terms when $z_2 \rightarrow z_3$. These facts indicate a staggered logarithmic module with three key

operators: Φ_0 , a weight 0 primary operator; $\partial\Phi_0$, the weight 1 primary descendant of Φ_0 ; and Φ_1 , the weight 1 logarithmic partner to $\partial\Phi_0$.

This structure is similar to the module $\mathfrak{J}_{1,4}$ discussed in [14], but appears for all values of κ , not just those with $c = 0$. Speculatively, this module may be related to the zero weight indicator operator used to gauge the position of a point relative to a given set of paths, e.g. in Schramm’s derivation of the SLE left passage probability [11].

The nature of the staggered module is such that we cannot fix a unique expression for the corresponding conformal block $G_{1,1}(1-x)$ because we are free to arbitrarily modify our choice by adding a multiple of $G_{-1,1}(1-x)$. We write an expression for the $x \approx 1$ blocks that includes this ambiguity. We do this by applying the logarithmic series expansion for the hypergeometric function, found for example in [19], to (13) and (14) and combining these blocks to have a leading term of unity

$$G_{1,1}(1-x) = A \frac{\Gamma(1-\kappa/4)\Gamma(2-\kappa/4)}{\Gamma(2-\kappa/2)} F_{1,1}(x) + (1-A) \frac{\Gamma(1+\kappa/4)\Gamma(\kappa/4)}{\Gamma(\kappa/2)} F_{3,1}(x), \quad (15)$$

and

$$G_{-1,1}(1-x) = (1-x)^{1-\kappa/8} {}_2F_1\left(\frac{4-\kappa}{4}, -\frac{\kappa}{4}; 2; 1-x\right). \quad (16)$$

We will discuss the ambiguity encoded by A again once we have obtained physical solutions for the correlation function.

We now introduce a convenient normalization for the conformal blocks

$$\begin{aligned} \mathcal{F}_{1,1}(x) &= F_{1,1}(x), \\ \mathcal{F}_{3,1}(x) &= \frac{\Gamma(1+\kappa/4)\Gamma(\kappa/4)\Gamma(2-\kappa/2)}{\Gamma(\kappa/2)\Gamma(1-\kappa/4)\Gamma(2-\kappa/4)} F_{3,1}(x), \\ \mathcal{G}_{1,1}(1-x) &= \frac{\Gamma(2-\kappa/2)}{\Gamma(1-\kappa/4)\Gamma(2-\kappa/4)} G_{1,1}(1-x), \quad \text{and} \\ \mathcal{G}_{-1,1}(1-x) &= \frac{\Gamma(1+\kappa/4)\Gamma(\kappa/4)}{\Gamma(\kappa/2-1)} G_{-1,1}(1-x). \end{aligned} \quad (17)$$

We choose $\mathcal{F}_{1,1}(x)$ so that we recover the disconnected limit

$$\langle \phi_{2,1}(z_1)\phi_{2,1}(z_2)\phi_{0,1}(z_2)\phi_{0,1}(z_2) \rangle \approx \langle \phi_{2,1}(z_1)\phi_{2,1}(z_2) \rangle \langle \phi_{0,1}(z_2)\phi_{0,1}(z_2) \rangle \quad (18)$$

when $|z_1 - z_2|$ and $|z_3 - z_4| \ll |z_1 - z_3|$. The other blocks are normalized to simplify the crossing relations, which become

$$\begin{aligned} \mathcal{F}_{1,1}(x) &= \mathcal{G}_{1,1}(1-x) + (1-A)\mathcal{G}_{-1,1}(1-x), \\ \mathcal{F}_{3,1}(x) &= \mathcal{G}_{1,1}(1-x) - A\mathcal{G}_{-1,1}(1-x), \\ \mathcal{G}_{1,1}(1-x) &= A\mathcal{F}_{1,1}(x) + (1-A)\mathcal{F}_{3,1}(x), \quad \text{and} \\ \mathcal{G}_{-1,1}(1-x) &= \mathcal{F}_{1,1}(x) - \mathcal{F}_{3,1}(x). \end{aligned} \quad (19)$$

We need to study the monodromy of our blocks as we move x around the points 0 and 1, in order to construct a single-valued physical solution. Moving clockwise around zero, the $x \approx 0$ blocks transform as

$$\begin{aligned} \mathcal{F}_{1,1}(x) &\xrightarrow{0\circlearrowright} \mathcal{F}_{1,1}(x), \\ \mathcal{F}_{3,1}(x) &\xrightarrow{0\circlearrowright} e^{-\kappa\pi i} \mathcal{F}_{3,1}(x). \end{aligned} \quad (20)$$

The effect of moving around 1 is more complicated due to the logarithm in the expansion of $\mathcal{G}_{1,1}(1-x)$ around $x = 1$. To determine the effect of these logarithmic terms we isolate

them from the remaining algebraic terms by applying the logarithmic series expansion to (15). We are careful to maintain our modified normalizations and after some simplifications involving gamma functions we find that

$$\mathcal{G}_{1,1}(1-x) = (1-x)^{-\kappa/8} S(1-x) + \frac{1}{2\pi} \tan(\pi\kappa/4) \ln(1-x) \mathcal{G}_{-1,1}(1-x), \quad (21)$$

where $S(1-x)$ is the regular power series contribution, which is regular about the point $x = 1$. On the other hand, taking x clockwise around 1 the logarithm transforms as

$$\ln(1-x) \xrightarrow{1\circlearrowright} \ln(1-x) - 2\pi i. \quad (22)$$

Thus moving clockwise around 1 we find that

$$\begin{aligned} \mathcal{G}_{1,1}(1-x) &\xrightarrow{1\circlearrowright} e^{\kappa\pi i/4} (\mathcal{G}_{1,1}(1-x) - i \tan(\pi\kappa/4) \mathcal{G}_{-1,1}(1-x)), \\ \mathcal{G}_{-1,1}(1-x) &\xrightarrow{1\circlearrowright} e^{\kappa\pi i/4} \mathcal{G}_{-1,1}(1-x), \end{aligned} \quad (23)$$

which can be combined with (19) yielding

$$\begin{aligned} \mathcal{F}_{1,1}(x) &\xrightarrow{1\circlearrowright} \sec\left(\frac{\kappa\pi}{4}\right) \mathcal{F}_{1,1}(x) + i e^{\kappa\pi i/4} \tan\left(\frac{\kappa\pi}{4}\right) \mathcal{F}_{3,1}(x), \\ \mathcal{F}_{3,1}(x) &\xrightarrow{1\circlearrowright} e^{\kappa\pi i/2} \sec\left(\frac{\kappa\pi}{4}\right) \mathcal{F}_{3,1}(x) - i e^{\kappa\pi i/4} \tan\left(\frac{\kappa\pi}{4}\right) \mathcal{F}_{1,1}(x). \end{aligned} \quad (24)$$

This completes our analysis of the chiral solution space. Now we need to sew together holomorphic and anti-holomorphic conformal blocks into a single-valued physical solution. Based on (20) we see that the physical solution must be of the form $\mathcal{F}_{1,1}(x)\mathcal{F}_{1,1}(\bar{x}) + C\mathcal{F}_{3,1}(x)\mathcal{F}_{3,1}(\bar{x})$. If we combine this with the condition that the correlation function remain unchanged when we move x around 1, then we can uniquely determine the physical solution $|\mathcal{F}_{1,1}(x)|^2 - |\mathcal{F}_{3,1}(x)|^2 = \mathcal{G}_{1,1}(1-x)\mathcal{G}_{-1,1}(1-\bar{x})$

$$+ \mathcal{G}_{-1,1}(1-x)\mathcal{G}_{1,1}(1-\bar{x}) + (1-2A)|\mathcal{G}_{-1,1}(1-x)|^2. \quad (25)$$

The negative coefficient serves as a reminder that we are working within a non-unitary theory.

As we expect, the ambiguity represented by A in (15) does not affect the form of our solution, which is easily seen since A is absent in the $x \approx 0$ blocks of (25). In terms of the $x \approx 1$ blocks the explicit appearance of A as a coefficient of $|\mathcal{G}_{-1,1}(1-x)|^2$ serves to cancel out the implicit dependence on A found in the $\mathcal{G}_{1,1}(1-x)$ block.

The non-diagonal combination of $x \approx 1$ blocks is another important feature of this solution, equivalent to a leading fusion term with nonzero spin. As $z_2 \rightarrow z_3$ the fusion to the logarithmic module with highest weight operator Φ_0 generates the block $\mathcal{G}_{1,1}(1-x)$, while the block $\mathcal{G}_{-1,1}(1-x)$ is generated by the regular subsector of this module with highest weight operator $\partial\Phi_0$. Thus, as $z_2 \rightarrow z_3$ the operator product expansion observed in this correlation function is

$$\phi_{2,1}(\mathbf{r} + \delta\mathbf{r})\phi_{0,1}(\mathbf{r}) = |\delta\mathbf{r}|^{-\kappa/4} (\delta\mathbf{r}) \cdot \nabla\Phi_0(\mathbf{r}) + O(|\delta\mathbf{r}|^{2-\kappa/4}), \quad (26)$$

where $\phi_{2,1}(\mathbf{r})$ and $\phi_{0,1}(\mathbf{r})$ represent respectively the factorized bulk twist and 2-leg operators used to construct this correlation function and $\Phi_0(\mathbf{r}) := \Phi_0(z)\Phi_0(\bar{z})$ is the scalar bulk version of the zero weight operator described above.

Restoring all relevant factors and the implicit dependence on the four coordinates we have $\langle\phi_{2,1}(z_1, \bar{z}_1)\phi_{2,1}(z_2, \bar{z}_2)\phi_{0,1}(z_3, \bar{z}_3)\phi_{0,1}(z_4, \bar{z}_4)\rangle$

$$\begin{aligned} &= \left| \frac{z_{21}}{z_{43}} \right|^2 \left| \frac{z_{43}z_{31}z_{42}}{z_{21}^3 z_{32}z_{41}} \right|^{\kappa/4} \left[\left| {}_2F_1\left(-\frac{\kappa}{4}, \frac{4-\kappa}{4}; \frac{4-\kappa}{2}; \frac{z_{21}z_{43}}{z_{31}z_{42}}\right) \right|^2 \right. \\ &\quad \left. - \frac{\Gamma\left(\frac{4+\kappa}{4}\right)^4 \Gamma\left(\frac{6-\kappa}{2}\right)^2}{\Gamma\left(\frac{2+\kappa}{2}\right)^2 \Gamma\left(\frac{8-\kappa}{4}\right)^4} \left| \frac{z_{21}z_{43}}{z_{31}z_{42}} \right|^{\kappa-2} \left| {}_2F_1\left(\frac{\kappa-4}{4}, \frac{\kappa}{4}; \frac{\kappa}{2}; \frac{z_{21}z_{43}}{z_{31}z_{42}}\right) \right|^2 \right]. \end{aligned} \quad (27)$$

2.2. Boundary $O(n)$ paths and twist operators

In the previous sections we found an expression for a simple bulk correlation function using 2-leg operators and twist operators. In this section we derive analogous correlation functions in regions with a boundary, anchoring paths to the boundary instead of loops in the bulk. In addition to the twist operators we use the boundary N -leg operators. These can be identified in the Coulomb gas as operators that change the boundary conditions by N steps within some neighborhood of their insertion. This leads to the identification with the boundary operator

$$h_{N\text{-leg}} = h_{1,N+1} = \frac{N(4 + 2N - \kappa)}{2\kappa}. \tag{28}$$

In particular we emphasize that the correlation functions we calculate in this section differ from the boundary correlation functions in [10] in that we only insert a single twist operator into our correlation functions. The effect of the twist operator in this case is to indicate the parity of loops separating the insertion point and boundary. If we have loops attached to the boundary the twist operator may have an unimportant ambiguity in the overall sign depending on which part of the boundary we choose to measure our parity from.

We first examine the correlation function which uses two 1-leg boundary operators to encode an SLE process from x_1 to x_2 in the presence of a twist operator. The correlation function takes the form

$$\langle \phi_{2,1}(z, \bar{z}) \phi_{1,2}(x_1) \phi_{1,2}(x_2) \rangle = (z - \bar{z})^{-2h_{2,1}} (x_2 - x_1)^{-2h_{1,2}} F \left(\frac{(z - \bar{z})(x_2 - x_1)}{(z - x_1)(x_2 - \bar{z})} \right), \tag{29}$$

and the null states of $\phi_{2,1}$ and $\phi_{1,2}$ imply that it is annihilated by

$$\frac{3\partial_{\bar{z}}^2}{2(1 + 2h_{2,1})} - \left(\frac{h_{2,1}}{(\bar{z} - z)^2} - \frac{\partial_{\bar{z}}}{\bar{z} - z} + \frac{h_{1,2}}{(x_1 - z)^2} - \frac{\partial_{x_1}}{x_1 - z} + \frac{h_{1,2}}{(x_2 - z)^2} - \frac{\partial_{x_2}}{x_2 - z} \right), \tag{30}$$

and

$$\frac{3\partial_{x_1}^2}{2(1 + 2h_{1,2})} - \left(\frac{h_{2,1}}{(z - x_1)^2} - \frac{\partial_z}{z - x_1} + \frac{h_{2,1}}{(\bar{z} - x_2)^2} - \frac{\partial_{\bar{z}}}{\bar{z} - x_2} + \frac{h_{1,2}}{(x_2 - x_1)^2} - \frac{\partial_{x_2}}{x_2 - x_1} \right), \tag{31}$$

respectively. Applying these differential operators to the correlation function and letting $\{\bar{z}, z, x_2, x_1\} \rightarrow \{0, x, 1, \infty\}$, we find two differential equations governing $F(x)$

$$0 = F''(x) + \frac{2(4 - \kappa) - (8 - \kappa)x}{4x(1 - x)} F'(x) - \frac{(6 - \kappa)}{8(1 - x)^2} F(x), \quad \text{and} \tag{32}$$

$$0 = F''(x) - \frac{2(4 - \kappa) - (4 - 2\kappa)x}{\kappa x(1 - x)} F'(x) - \frac{(3\kappa - 8)}{4\kappa(1 - x)^2} F(x). \tag{33}$$

The only common solution to both of these differential equations is

$$F(x) = \frac{2 - x}{2\sqrt{1 - x}}. \tag{34}$$

In the strip $\mathbb{S} = \{s = u + vi \mid u \in \mathbb{R}, 0 < v < \pi\}$ this correlation function takes on a simple form. We map the two anchoring points to $\pm\infty$ using

$$s(z) = \log \left(\frac{z - x_1}{z - x_2} \right), \tag{35}$$

so that the cross ratio is

$$x = \frac{(z - \bar{z})(x_2 - x_1)}{(z - x_1)(x_2 - \bar{z})} = 1 - \frac{(\bar{z} - x_1)(x_2 - z)}{(z - x_1)(x_2 - \bar{z})} = 1 - e^{2iv}. \quad (36)$$

Thus, in the strip the correlation function can be written

$$\langle \phi_{2,1}(s, \bar{s}) \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}} = \langle \phi_{2,1}(s, \bar{s}) \rangle_{\mathbb{S}} \langle \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}} \cos(v), \quad (37)$$

which is independent of our choice of κ .

We also include the correlation function describing a combination of a twist operator and two 2-leg boundary operators. The two paths generated by the 2-leg operators can be thought of as a double SLE: two SLE processes starting from a common point and driven toward a common point while conditioned not to meet. This correlation function is of the form

$$\langle \phi_{2,1}(z, \bar{z}) \phi_{1,3}(x_1) \phi_{1,3}(x_2) \rangle = (z - \bar{z})^{-2h_{2,1}} (x_2 - x_1)^{-2h_{1,3}} F \left(\frac{(z - \bar{z})(x_2 - x_1)}{(z - x_1)(x_2 - \bar{z})} \right), \quad (38)$$

and the null states of $\phi_{2,1}$ and $\phi_{1,3}$ imply that it is annihilated by the differential operators

$$\frac{3\partial_z^2}{2(1 + 2h_{2,1})} - \left(\frac{h_{2,1}}{(\bar{z} - z)^2} - \frac{\partial_{\bar{z}}}{\bar{z} - z} + \frac{h_{1,2}}{(x_1 - z)^2} - \frac{\partial_{x_1}}{x_1 - z} + \frac{h_{1,2}}{(x_2 - z)^2} - \frac{\partial_{x_2}}{x_2 - z} \right), \quad (39)$$

and

$$\begin{aligned} \frac{\partial_{x_1}^3}{h_{1,3}(1 + h_{1,3})} - \frac{2}{h_{1,3}} \left(\frac{h_{2,1}\partial_{x_1}}{(z - x_1)^2} - \frac{\partial_z\partial_{x_1}}{z - x_1} + \frac{h_{2,1}\partial_{x_1}}{(\bar{z} - x_2)^2} - \frac{\partial_{\bar{z}}\partial_{x_1}}{\bar{z} - x_2} + \frac{h_{1,3}\partial_{x_1}}{(x_2 - x_1)^2} \right. \\ \left. - \frac{\partial_{x_2}\partial_{x_1}}{x_2 - x_1} \right) - \left(\frac{2h_{2,1}}{(z - x_1)^3} - \frac{\partial_z}{(z - x_1)^2} + \frac{2h_{2,1}}{(\bar{z} - x_2)^3} - \frac{\partial_{\bar{z}}}{(\bar{z} - x_2)^2} \right. \\ \left. + \frac{h_{1,3}}{(x_2 - x_1)^3} - \frac{\partial_{x_2}}{(x_2 - x_1)^2} \right), \end{aligned} \quad (40)$$

respectively. The corresponding differential equations are

$$0 = F''(x) + \frac{2(4 - \kappa) - (8 - \kappa)x}{4x(1 - x)} F'(x) - \frac{8 - \kappa}{4(1 - x)^2} F(x), \quad \text{and} \quad (41)$$

$$\begin{aligned} 0 = F'''(x) - \frac{2(16 - 3\kappa + (3\kappa - 8)x)}{\kappa x(1 - x)} F''(x) \\ + \frac{6(8 - \kappa)(4 - \kappa) + 4(6 - \kappa)(3\kappa - 8)x - (8 - \kappa)(3\kappa - 8)x^2}{\kappa^2 x^2(1 - x)^2} F'(x) \\ + \frac{(3\kappa - 8)(8 - \kappa)(2 - x)}{\kappa^2 x(1 - x)^3} F(x), \end{aligned} \quad (42)$$

which only have one common solution

$$F(x) = 1 + \frac{8 - \kappa}{6 - \kappa} \frac{x^2}{4(1 - x)}. \quad (43)$$

In the strip geometry this becomes

$$\begin{aligned} \langle \phi_{2,1}(s, \bar{s}) \phi_{1,3}(-\infty) \phi_{1,3}(\infty) \rangle_{\mathbb{S}} \\ = \langle \phi_{2,1}(s, \bar{s}) \rangle_{\mathbb{S}} \langle \phi_{1,3}(-\infty) \phi_{1,3}(\infty) \rangle_{\mathbb{S}} \left(1 - \frac{8 - \kappa}{6 - \kappa} \sin^2(v) \right). \end{aligned} \quad (44)$$

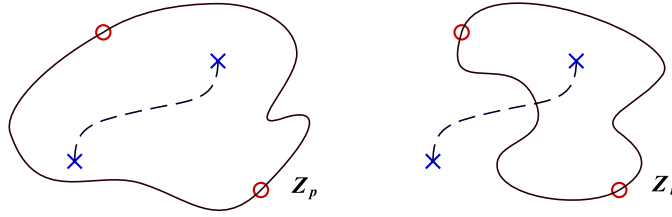


Figure 1. The two topological weights for a pair of 2-leg operators (circles; red online) and a pair of twist operators (crosses; blue online) at $n = 0$.

3. The case $n = 0$: self-avoiding loops

In the previous section we found expressions for several $O(n)$ correlation functions with $n \in (-2, 2)$. Now take $n = 0$; in this limit the $O(n)$ loop model corresponds to self-avoiding walks and loops and has corresponding SLE/CLE parameter $\kappa = 8/3$. With $n = 0$ all loops are suppressed and the only contribution to the partition function is the empty set so that $Z = 1$. In [10] this was circumnavigated by taking a small n expansion and keeping the first-order term, which was equivalent to conditioning the configurations on the existence of one loop. While we need to do the same thing in spirit, our job is made trivial by the inclusion of 1- or 2-leg operators. As we discussed above the loops associated with these operators have unit weight and are not suppressed even at $n = 0$. With $n = 0$ the weight of the twist operator is $h_{\text{twist}} = 0$. As we will see, this reflects the fact that, in the absence of background loops, the twist operator has a similar interpretation to Schramm’s indicator operator.

3.1. The anchored self-avoiding loop with twist operators

We return to the bulk correlation function with two 2-leg operators and two twist operators. Because of the 2-leg operators with weight $h_{2\text{-leg}} = 1/3$, at $n = 0$ we are restricted to those configurations with a self-avoiding loop (SAL) anchored at z_3 and z_4 . The total weight of these allowed configurations is given by the two point function

$$Z = \langle \phi_{0,1}(z_3, \bar{z}_3) \phi_{0,1}(z_4, \bar{z}_4) \rangle = |z_{34}|^{-4/3}. \tag{45}$$

With the inclusion of the twist operators we can further decompose Z into two separate weights based on how the anchored SAL interacts with the twist defect. The two relevant possibilities are illustrated in figure 1. The weight of configurations in which the anchored loop segregates the two twist defects is denoted by Z_n . The weight of configurations where the loop fails to separate the twist defects is Z_p . The index refers to whether the weight of the contribution is positive or negative with respect to the twist defect.

Now in terms of these weights it should be apparent that

$$Z = Z_p + Z_n, \quad \text{and} \tag{46}$$

$$Z_{\text{twist}} = Z_p - Z_n, \tag{47}$$

with the twist partition function given by letting $\kappa = 8/3$ in (27)

$$Z_{\text{twist}} = \left| \frac{z_{31}z_{42}}{z_{43}^2 z_{32}z_{41}} \right|^{2/3} \left[\left| {}_2F_1 \left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}; \frac{z_{21}z_{43}}{z_{31}z_{42}} \right) \right|^2 - \frac{\Gamma(5/3)^6}{\Gamma(4/3)^4 \Gamma(7/3)^2} \left| \frac{z_{21}z_{43}}{z_{31}z_{42}} \right|^{2/3} \left| {}_2F_1 \left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; \frac{z_{21}z_{43}}{z_{31}z_{42}} \right) \right|^2 \right]. \tag{48}$$

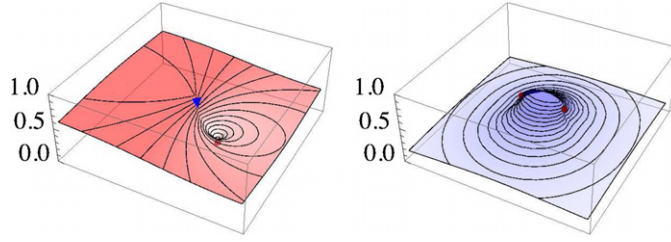


Figure 2. On the left is a plot of $P_n(0, \infty; 1, \bullet)$, on the right is a plot of $P_n(\bullet, \infty; 1, -1)$. Circles (red online) indicate the positions of the 2-leg operators. The triangle (blue online) marks the position of the twist defect, which sits away from infinity.

This allows us to determine the probability that a SAL separates z_1 and z_2 in the upper half-plane, given that the loop passes through z_3 and z_4

$$\begin{aligned}
 P_n(z_1, z_2; z_3, z_4) &= \frac{Z_n}{Z} = \frac{Z - Z_{\text{twist}}}{2Z} \\
 &= \frac{1}{2} \left[1 - \left| \frac{1}{1-x} \right|^{2/3} \left| {}_2F_1 \left(-\frac{2}{3}, \frac{1}{3}; \frac{2}{3}; x \right) \right|^2 \right. \\
 &\quad \left. + \frac{\Gamma(5/3)^6}{\Gamma(4/3)^4 \Gamma(7/3)^2} \left| \frac{x}{1-x} \right|^{2/3} \left| {}_2F_1 \left(-\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; x \right) \right|^2 \right], \tag{49}
 \end{aligned}$$

in terms of the cross ratio $x = z_{21}z_{43}/z_{31}z_{42}$. Plots of this probability are included in figure 2.

As x goes to 0 the twist defects and loop are well separated and the probability that the loop segregates the twist defects also goes to zero. Maximum value for P_n occurs when cross ratio goes to 2. Because of global conformal symmetry we are free to fix the position of three of our operators. If we place the two 2-leg operators at ± 1 and one of the twist operators at infinity, then placing the second twist operator at the origin should maximize the probability of sitting inside the anchored loop, this corresponds to $x = 2$. The value of P_n for $x = 2$ is

$$P_n^{\text{max}} = \frac{1}{2} + \frac{9\Gamma(5/6)^6}{4\pi^3} \approx 0.6501 \dots \tag{50}$$

As a final comment for $n = 0$ we note that the weight Z_p is equal to a related correlation function with the twists replaced by magnetization operators. Pairs of magnetization operators behave in a fashion akin to pairs of twist operators; they change the weight of loops that separate the two points of insertion. But whereas twist operators take $n \rightarrow -n$ the magnetization operators take $n \rightarrow 0$. Since there can be no loop (or cluster hull) which separates them from each other, the magnetization operators measure configurations where the two points reside in the same cluster, which at $n = 0$ is exactly P_p . The magnetization operator is $\phi_{1/2,0}$ [1] and

$$\langle \phi_{1/2,0}(z_1, \bar{z}_1) \phi_{1/2,0}(z_2, \bar{z}_2) \phi_{0,1}(z_3, \bar{z}_3) \phi_{0,1}(z_4, \bar{z}_4) \rangle = Z[1 - P_n(z_1, z_2; z_3, z_4)]. \tag{51}$$

For $n = 0$ the magnetization operator has conformal weight $h_{1/2,0} = 0 = h_{2,1}$. Of course, the fact that these weights are equal is a necessary condition for their correlation functions to have common solutions.

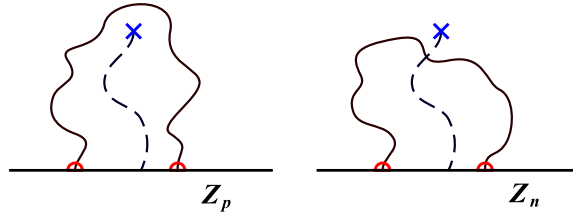


Figure 3. The two topological weights of the upper half-plane $SLE_{8/3}$ (semi-circles; red online) with respect to the twist defect (cross; blue online).

3.2. $SLE_{8/3}$ with a twist operator

Now we take the correlation function $\langle \phi_{2,1}(z, \bar{z})\phi_{1,2}(x_1)\phi_{1,2}(x_2) \rangle$ and restrict ourselves to $n = 0$ and therefore to an $SLE_{8/3}$ process. We condition our correlation functions upon the existence of an SLE path from x_1 to x_2 so that the partition function is

$$Z = Z_p + Z_n = \langle \phi_{1,2}(x_1)\phi_{1,2}(x_2) \rangle = (x_2 - x_1)^{-5/4}, \quad (52)$$

dividing the ensemble of configurations that contribute to this partition function into two parts as in figure 3.

Because the twist operators are zero weight operators we may rewrite the result in the strip (37) as

$$\frac{\langle \phi_{2,1}(s, \bar{s})\phi_{1,2}(-\infty)\phi_{1,2}(\infty) \rangle_{\mathbb{S}}}{\langle \phi_{1,2}(-\infty)\phi_{1,2}(\infty) \rangle_{\mathbb{S}}} = \frac{Z_p - Z_n}{Z} = \cos v, \quad (53)$$

the twist operator correlation function conditioned on the $SLE_{8/3}$. The probability that the SLE separates the twist operator from the bottom edge of the strip is

$$P_p(v) = \frac{Z_p}{Z} = \frac{1 + \cos v}{2} = \cos^2(v/2). \quad (54)$$

As we expect, this is Schramm’s formula for the left crossing probability in the strip with $\kappa = 8/3$ [11].

3.3. Double $SLE_{8/3}$ with a twist operator

We now reexamine the correlation function $\langle \phi_{2,1}(z, \bar{z})\phi_{1,3}(x_1)\phi_{1,3}(x_2) \rangle$ for the value $n = 0$. Conditioning the system on the existence of a double $SLE_{8/3}$ we have partition function

$$Z = Z_{p1} + Z_n + Z_{p2} = \langle \phi_{1,3}(x_1)\phi_{1,3}(x_2) \rangle = (x_2 - x_1)^{-4}, \quad (55)$$

and the total mass of these configurations may be decomposed into three parts as in figure 4.

Our previous result (44) tells us that in the strip

$$\frac{\langle \phi_{2,1}(s, \bar{s})\phi_{1,3}(-\infty)\phi_{1,3}(\infty) \rangle_{\mathbb{S}}}{\langle \phi_{1,3}(-\infty)\phi_{1,3}(\infty) \rangle_{\mathbb{S}}} = \frac{Z_{p1} - Z_n + Z_{p2}}{Z} = 1 - \frac{8}{5} \sin^2(v), \quad (56)$$

from which we determine the probability that the twist operator lies between the two SLEs

$$P_n(v) = \frac{Z_n}{Z} = \frac{4}{5} \sin^2(v). \quad (57)$$

This matches the extension of Schramm’s formula to double SLEs derived in [12] for $\kappa = 8/3$. We are unable to disentangle the contributions from Z_{p1} and Z_{p2} using twist operators because of their parity symmetry.

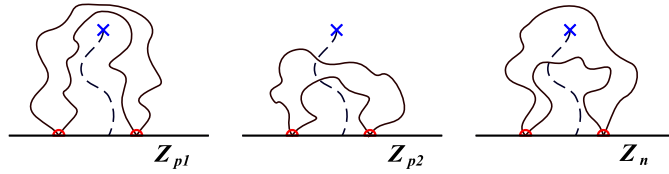


Figure 4. The three topological weights of the upper half-plane double $SLE_{8/3}$ (semi-circles; red online) with respect to the twist defect (cross; blue online).

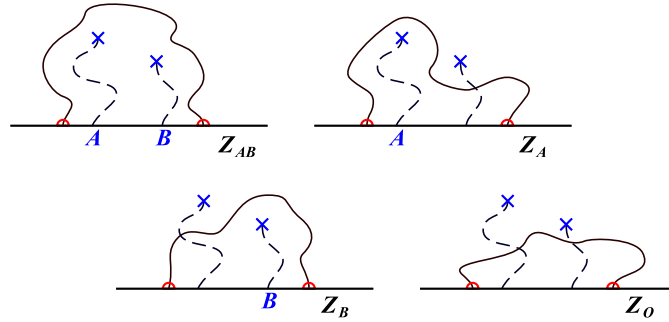


Figure 5. The topological weights of the upper half-plane $SLE_{8/3}$ (semi-circles; red online) with respect to the twist defects at z_A and z_B (crosses; blue online).

In this section and the last we use twist operators to derive two $SLE_{8/3}$ results having been previously derived using zero-weight indicator operators. As we have observed in the $O(n = 0)$ model the twist operator is simply related to the indicator operator, aside from issues of loop number parity.

4. A new $SLE_{8/3}$ result

In the last section we showed that at $\kappa = 8/3$ the twist operator can be used in place of Schramm’s zero weight indicator operator to derive various probabilities. In this section we describe how twist operators can be used to calculate a new $SLE_{8/3}$ result. We will use the null-state of the twist operator as an extra condition to determine a correlation function with one path; with a single path there will be no parity issues and the twist operator should play the role of the indicator operator perfectly. We decompose the total weight of all SLE paths in the infinite strip according to the relative winding around two points, labeled by A and B in figure 5. We denote these weights by Z_i , where the index i labels which of A and/or B are adjacent to the bottom edge of the strip.

We can write down four possible correlation functions based on whether or not we place twist operators at z_A and/or z_B . We fix all signs so that the twists parity is measured with respect to the bottom edge. The four correlation functions and their expressions in terms of our decomposition are

$$\frac{Z_{AB} + Z_A + Z_B + Z_O}{Z} = \frac{\langle \phi_{1,2}(-\infty)\phi_{1,2}(\infty) \rangle_S}{\langle \phi_{1,2}(-\infty)\phi_{1,2}(\infty) \rangle_S} = 1, \tag{58}$$

$$\frac{Z_{AB} + Z_A - Z_B - Z_O}{Z} = \frac{\langle \phi_{2,1}(s_A, \bar{s}_A)\phi_{1,2}(-\infty)\phi_{1,2}(\infty) \rangle_S}{\langle \phi_{1,2}(-\infty)\phi_{1,2}(\infty) \rangle_S} = \cos(v_A), \tag{59}$$

$$\frac{Z_{AB} - Z_A + Z_B - Z_O}{Z} = \frac{\langle \phi_{2,1}(s_B, \bar{s}_B) \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}}}{\langle \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}}} = \cos(v_B), \quad (60)$$

$$\begin{aligned} \frac{Z_{AB} - Z_A - Z_B + Z_O}{Z} &= \frac{\langle \phi_{2,1}(s_A, \bar{s}_A) \phi_{2,1}(s_B, \bar{s}_B) \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}}}{\langle \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}}} \\ &= \cos(v_A) \cos(v_B) + \sin(v_A) \sin(v_B) \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3}; 1 - \sigma \right) \right], \end{aligned} \quad (61)$$

with the parameter σ defined as

$$\sigma = \frac{\cosh(u_B - u_A) - \cos(v_A - v_B)}{\cosh(u_B - u_A) - \cos(v_A + v_B)}. \quad (62)$$

Of these equations the first three are either trivial or equivalent to (54). However, equation (61) is new, and can be shown to be the unique function which obeys both the boundary $\phi_{1,2}$ null-state and bulk $\phi_{2,1}$ null-state differential equations, while satisfying the limiting condition

$$\lim_{u_2 - u_1 \rightarrow \infty} \frac{\langle \phi_{2,1}(s_A, \bar{s}_A) \phi_{2,1}(s_B, \bar{s}_B) \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}}}{\langle \phi_{1,2}(-\infty) \phi_{1,2}(\infty) \rangle_{\mathbb{S}}} = \cos(v_A) \cos(v_B). \quad (63)$$

We outline the derivation of this new quantity beginning with the form implied by conformal symmetry in the upper half-plane

$$\langle \phi_{2,1}(z_A, \bar{z}_A) \phi_{2,1}(z_B, \bar{z}_B) \phi_{1,2}(x_1) \phi_{1,2}(x_2) \rangle = (x_2 - x_1)^{-5/4} \lambda \mu F(\lambda, \mu, \nu), \quad (64)$$

with variables chosen as functions of cross ratios that simplify the dependence on the strip variables

$$\lambda = \cos(v_A) = \frac{(z_A - x_1)(x_2 - \bar{z}_A) + (\bar{z}_A - x_1)(x_2 - z_A)}{2|z_A - x_1||x_2 - z_A|}, \quad (65)$$

$$\mu = \cos(v_B) = \frac{(z_B - x_1)(x_2 - \bar{z}_B) + (\bar{z}_B - x_1)(x_2 - z_B)}{2|z_B - x_1||x_2 - z_B|}, \quad \text{and} \quad (66)$$

$$\nu = \cosh(u_B - u_A) = \frac{|z_A - x_1|^2 |x_2 - z_B|^2 + |z_B - x_1|^2 |x_2 - z_A|^2}{2|z_B - x_1||x_2 - z_A||z_A - x_1||x_2 - z_B|}. \quad (67)$$

Each operator in the correlation function has a second-order null state, leading to six partial differential equations for $F(\lambda, \mu, \nu)$. These equations are cumbersome and we do not record them here. However, they can be rearranged to yield

$$0 = \lambda(1 - \lambda^2) \partial_\lambda F(\lambda, \mu, \nu) - \mu(1 - \mu^2) \partial_\mu F(\lambda, \mu, \nu) - \nu(\lambda^2 - \mu^2) \partial_\nu F(\lambda, \mu, \nu). \quad (68)$$

If we make the change of variables

$$\rho = \lambda \mu = \cos(v_A) \cos(v_B), \quad (69)$$

$$\sigma = \frac{\nu - \lambda \mu - \sqrt{(1 - \lambda^2)(1 - \mu^2)}}{\nu - \lambda \mu + \sqrt{(1 - \lambda^2)(1 - \mu^2)}} = \frac{\cosh(u_B - u_A) - \cos(v_B - v_A)}{\cosh(u_B - u_A) - \cos(v_B + v_A)}, \quad (70)$$

$$\tau = \frac{\sqrt{(1 - \lambda^2)(1 - \mu^2)}}{\lambda \mu} = \tan(v_A) \tan(v_B), \quad (71)$$

then (68) implies $0 = \partial_\rho F(\rho, \sigma, \tau)$ indicating a reduction in the effective number of variables

$$F(\lambda, \mu, \nu) = H(\sigma, \tau). \quad (72)$$

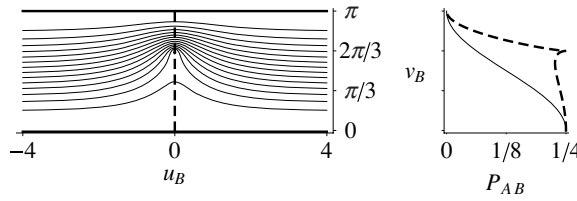


Figure 6. On the left is a contour plot of P_{AB} as a function of $s_B = u_B + v_B i$ in the strip, given that $s_A = 2\pi i/3$. On the right is the detail of the function at $u_B = 0$ (dashed thick line) and the asymptotic value as $u_B \rightarrow \infty$ (thin line).

In terms of this new function the original differential equations are equivalent to

$$0 = (1 - \sigma)\tau H(\sigma, \tau) - \tau(1 + \tau + \sigma - \sigma\tau)\partial_\tau H(\sigma, \tau) + 3\sigma(1 - \sigma)\partial_\sigma H(\sigma, \tau), \quad \text{and} \quad (73)$$

$$0 = \partial_\tau^2 H(\sigma, \tau), \quad (74)$$

with two-dimensional solution space

$$H(\sigma, \tau) = c_1 \left\{ 1 + \tau \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right] \right\} + c_2 \tau \left(\frac{\sigma}{(1 - \sigma)^2} \right)^{1/3}. \quad (75)$$

To pick out the physical solution we enforce the limit (63), or equivalently

$$\lim_{\sigma \rightarrow 1} H(\sigma, \tau) = 1. \quad (76)$$

Since the second term diverges in this limit we require that $c_2 = 0$ and (61) is the result of setting $c_1 = 1$ and restoring the other factors to the correlation function.

With (58)–(61) we can construct the probabilities $P_i = Z_i/Z$ for the winding configurations in figure 5

$$P_{AB} = \cos^2(v_A/2) \cos^2(v_B/2) + \frac{\sin(v_A) \sin(v_B)}{4} \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right], \quad (77)$$

$$P_A = \cos^2(v_A/2) \sin^2(v_B/2) - \frac{\sin(v_A) \sin(v_B)}{4} \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right], \quad (78)$$

$$P_B = \sin^2(v_A/2) \cos^2(v_B/2) - \frac{\sin(v_A) \sin(v_B)}{4} \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right], \quad (79)$$

$$P_O = \sin^2(v_A/2) \sin^2(v_B/2) + \frac{\sin(v_A) \sin(v_B)}{4} \left[1 - \sigma {}_2F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right]. \quad (80)$$

A typical example of the probability P_{AB} in a strip is plotted in figure 6.

These probabilities are conformally invariant and apply to the upper half-plane as well as in the strip. If the SLE in the upper half-plane runs from the origin to infinity then we only need to know that $v_{A(B)}$ is the argument of $z_{A(B)}$ and that in the half-plane

$$\sigma = \frac{(z_B - z_A)(\bar{z}_B - \bar{z}_A)}{(z_B - \bar{z}_A)(\bar{z}_B - z_A)} = \frac{r_B^2 - 2r_A r_B \cos(v_B - v_A) + r_A^2}{r_B^2 - 2r_A r_B \cos(v_B + v_A) + r_A^2}. \quad (81)$$

5. Logarithmic conformal field theory

In the last section we saw that the chiral correlation function $\langle \phi_{2,1} \phi_{2,1} \phi_{2,1} \phi_{2,1} \phi_{1,2} \phi_{1,2} \rangle$ allows two linearly independent solutions. These solutions are non-trivial and consistently satisfy all of the null-state conditions of the $\phi_{1,2}$ and $\phi_{2,1}$ operators. To understand why this is significant we recall the discussions of [14, 16] involving the logarithmic structure of the $c = 0$ CFT generated by $\phi_{2,1}$ and raising the question of whether this CFT is compatible with that generated by $\phi_{1,2}$. Specifically we note that in our physical interpretation of this correlation function, the two different species of operators appear as either bulk or boundary operators, but never as both. This allows a resolution of the co-existence issue and an explanation of the disagreement between [14] and [16] predicting which logarithmic partner appears in the self-avoiding walk and percolation models.

In what follows note that our Kac indices are in reverse order to those in [14] because we use the $SLE_{8/3}$ convention while they work with the dual SLE_6 convention.

We begin with the sector of our CFT generated by the Verma module $\mathfrak{M}_{2,1}$ of which $\phi_{2,1}$ with weight $h_{2,1} = 0$ is the highest weight state. Fusing this module with itself leads to a variety of descendant modules

$$\mathfrak{M}_{2,1} \times \mathfrak{M}_{2,1} = \mathfrak{M}_{1,1} + \mathfrak{M}_{3,1}, \tag{82}$$

$$\mathfrak{M}_{2,1} \times \mathfrak{M}_{2,1} \times \mathfrak{M}_{2,1} = \mathfrak{M}_{2,1} + \mathfrak{I}_{4,1}, \tag{83}$$

$$\mathfrak{M}_{2,1} \times \mathfrak{M}_{2,1} \times \mathfrak{M}_{2,1} \times \mathfrak{M}_{2,1} = \mathfrak{M}_{1,1} + 3\mathfrak{M}_{3,1} + \mathfrak{I}_{5,1}. \tag{84}$$

Most notably we are interested in the staggered logarithmic module $\mathfrak{I}_{5,1}$ which contains $\phi_{5,1}$, a candidate for the logarithmic partner to the stress energy tensor.

The other potential logarithmic partner $\phi_{1,3}$, occurs in the sector generated by $\mathfrak{M}_{1,2}$ with highest weight $h_{1,2} = 5/8$,

$$\mathfrak{M}_{1,2} \times \mathfrak{M}_{1,2} = \mathfrak{I}_{1,3}. \tag{85}$$

Each of the staggered modules $\mathfrak{I}_{1,3}$ and $\mathfrak{I}_{5,1}$ represent two distinct, but overlapping, conformal channels. The logarithmic channel includes contributions from the identity $\mathbf{1}$, the stress tensor T , and the logarithmic partner to the stress tensor $\phi := \phi_{1,3}$ or $\phi_{5,1}$. The regular channel only contains contributions from T and its descendants.

Conformal symmetry dictates that under the action of the Virasoro generators

$$L_0|\phi\rangle = 2|\phi\rangle + |T\rangle, \quad L_1|\phi\rangle = 0, \quad L_2|\phi\rangle = \beta|\mathbf{1}\rangle, \tag{86}$$

and fixes the forms of the two point functions,

$$\langle \phi(\xi)T(0) \rangle = \langle T(\xi)\phi(0) \rangle = \beta/\xi^4, \quad \text{and} \quad \langle T(\xi)T(0) \rangle = 0. \tag{87}$$

There is also a type of gauge symmetry associated with the logarithmic partner because we can let $\phi \rightarrow \phi + CT$ without changing the algebraic properties of ϕ . This symmetry raises questions when two logarithmic families are fused, but plays no role in the quantities discussed here.

The mixed two point function $\langle \phi_{1,3}\phi_{5,1} \rangle$ is undefinable. It cannot be consistently normalized because the two constants $\beta_{1,3} = 5/6$ and $\beta_{5,1} = -5/8$ are not the same [16]. Thus $\mathfrak{M}_{2,1}$ and $\mathfrak{M}_{1,2}$ cannot both be included in a chiral theory; such a theory would contain undefinable objects from $\mathfrak{I}_{1,3} \times \mathfrak{I}_{5,1}$. This means that there can be no pure bulk or boundary theory containing both modules, but does not necessarily eliminate mixed cases, like the present one, where one module resides in the bulk and the other in the boundary theory.

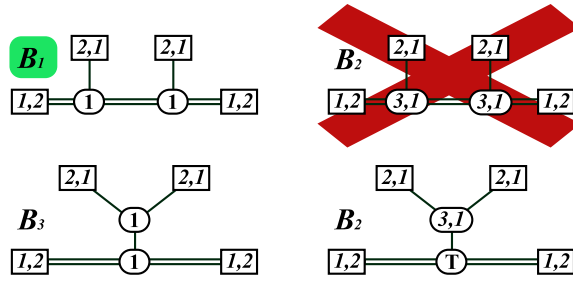


Figure 7. Conformal blocks with respect to bulk–boundary fusion. Rectangles denote operators and ovals denote fusion products. Boundary operators sit on double lines with bulk operators above. The block with incompatible boundary operators is marked with an ‘X’. The physical block is highlighted (green online).

We now identify the fusion channels associated with the conformal blocks of our correlation function. The two conformal blocks in (75) are defined about $\sigma = 1$. We can identify the blocks about $\sigma = 0$ using the crossing symmetry relation

$$\left[\rho + \rho\tau \left(1 - \sigma_2 F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| 1 - \sigma \right) \right) \right]_{B_1} = \left[\rho + \rho\tau \left(1 + \sigma_2 F_1 \left(1, \frac{4}{3}; \frac{5}{3} \middle| \sigma \right) \right) \right]_{B_3} - \left[\frac{\Gamma(5/3)\Gamma(2/3)\rho\tau\sigma^{1/3}}{\Gamma(4/3)(1 - \sigma)^{2/3}} \right]_{B_2}. \quad (88)$$

We name the blocks according to the bracket labels in (88).

The limit $\sigma \rightarrow 1$ can be achieved by taking both bulk operators to distinct boundary points. Series expansions in orders of v_A and v_B show that B_1 and B_2 are conformal blocks with exponent $0 = h_{1,1}$ and $1/3 = h_{3,1}$, respectively. So B_1 is the conformal block with the bulk operators fused to the boundary via the identity channel and B_2 is the block with the bulk operators fused to the boundary via $\phi_{3,1}$. Since a boundary theory cannot support both $\phi_{3,1}$ and $\phi_{1,2}$, the B_2 block is problematic, as we will discuss shortly. These blocks are represented graphically in the first row of figure 7.

On the other hand, the limit $\sigma = 0$ corresponds to fusing the bulk operators and subsequently bringing the fusion product to the boundary. We identify the conformal blocks in this limit by first expanding in orders of $|s_B - s_A| \ll v_A$ and then in orders of v_A . We find that B_3 and B_2 are the conformal blocks in this limit with leading terms proportional to $|s_B - s_A|^0 v_A^0$ and $|s_B - s_A|^{2/3} v_A^{4/3}$, respectively. This implies that block B_3 represents the fusion of the bulk operators to the identity, and the subsequent fusion to the boundary.

For blocks B_2 , the leading $|s_B - s_A|$ exponent is $2/3 = 2h_{3,1}$, so the twist operators fuse to a $\phi_{3,1}$. The leading v_A exponent of $4/3 = 2 - 2h_{3,1}$ then indicates that the subsequent bulk–boundary fusion is via a weight 2 operator. Of the fusion channels available to the $\phi_{3,1}$ fusion product only the regular channel of $\mathfrak{J}_{5,1}$ has a weight 2 highest weight operator, namely the stress tensor. Thus block B_2 represents the fusion of the bulk operators through the $\phi_{3,1}$ channel and the subsequent fusion to the boundary via the T channel. Both of these blocks are represented graphically in the second row of figure 7.

We now return to the B_2 block, which we observed is poorly defined due to the inadmissible bulk–boundary fusion channel as $v_A, v_B \rightarrow 0$. In order to ensure consistency we must exclude B_2 from the set of $\sigma \rightarrow 1$ conformal blocks. Instead of a two-dimensional solution space we have only one solution, the block B_1 ; an encouraging prospect since this is precisely the physical solution we identified in section 4. Interestingly, B_2 is not excluded from the set of

$\sigma \rightarrow 0$ conformal blocks, but instead appears only in the combination $B_3 - B_2 = B_1$. Thus it seems that the bulk fusion channel is fixed so that

$$\phi_{2,1}(\mathbf{r} + \delta\mathbf{r})\phi_{2,1}(\mathbf{r}) = \mathbf{1}(\mathbf{r}) - C|\delta\mathbf{r}|^{2/3}\phi_{3,1}(\mathbf{r}) + O(|\delta\mathbf{r}|^{5/3}), \quad (89)$$

with C a fixed OPE coefficient. If they were to be taken on their own the blocks B_2 or B_3 would be divergent in the limit where $u_1 - u_2 \rightarrow \infty$, thus their coupling seems natural.

The correlation function of an anchored loop with twist operators given in (25) was, $|\mathcal{F}_{1,1}(x)|^2 - |\mathcal{F}_{3,1}(x)|^2$. This is also the identity block minus the $\phi_{3,1}$ block in agreement with (89). At $n = 0$ the energy density operator $\phi_{3,1}$ is sensitive to local loops, acting as a 2-leg operator. Thus (89) is the natural choice for the OPE of two twist operators: the identity gives weight 1 to all the configurations except those where a path separates the twist operators and these get a negative weight due to the $\phi_{3,1}$ term.

Now that we understand how the bulk operators couple to the boundary theory, we take a minute to think of the boundary operators. The bulk operators fuse to the boundary through the identity and stress tensor. For these blocks to be non-trivial the fusion of the $\phi_{1,2}$ operators must contain $\mathbf{1}$ and/or $\phi_{1,3}$, thus occurring through the logarithmic channel. Even the two point function on which we condition our SLE_{8/3} requires the logarithmic fusion channel. This reinforces the assertion, made in [14], that the boundary LCFT describing self-avoiding loops should contain the logarithmic partner $\phi_{1,3}$ with parameter $\beta_{1,3} = 5/6$.

While our results regarding the boundary theory are in agreement with the predictions of Mathieu and Ridout, these predictions seem to contradict the earlier assertion of Gurarie and Ludwig in [16] that the logarithmic partner to the stress tensor for the self-avoiding walk is $\phi_{5,1}$. However, Gurarie and Ludwig made their assertion based on correlation functions of bulk operators. In fact the results of [10] support the assertion of Gurarie and Ludwig. This is because the conformal blocks used to assemble the correlation function in [10] are equivalent to those appearing in the derivation for Cardy's and Watts' formulae, and it was argued in [14] that the appearance of these blocks requires the inclusion of $\phi_{5,1}$ in the corresponding CFT.

Thus it seems that we desire a boundary LCFT with two logarithmic partners of $T(z)$: $\phi_{5,1}$ in the bulk, and $\phi_{1,3}$ on the boundary. This is possible if the bulk operators fuse to the boundary exclusively via the identity module, as is the case for the three allowed conformal blocks in figure 7. Specifically this is a statement about the bulk–boundary OPE for SLE type conformal boundary conditions upon which $\phi_{1,2}$ can appear as a boundary condition changing operator, it is possible that other boundary conditions exist which do not support $\phi_{1,2}$ and which allow a fuller bulk–boundary fusion content.

5.1. The implications of this result on $\kappa = 6$ and critical percolation

In the case discussed above the bulk theory coupled to the boundary by the logarithmic boundary channel with $\phi_{1,3}$. We may consider the consequence of insisting that fusions instead occur through the $\phi_{5,1}$ channel. As we said in the previous section, without the $\phi_{1,3}$ channel the B_1 block must be excluded, leaving B_2 as the only solution. This alternate case is speculated to correspond to critical percolation [14, 16], and we point out that this chiral correlation function does indeed have a physical interpretation in this model.

In the remainder of this section we work in the SLE₆ convention so that the Kac indices in this section are reversed from those in the rest of the paper. Thus, in the percolation model the four zero weight operators correspond to SLE operators $\phi_{1,2}$, and the first element of the correlation function is a set of four paths attached to the boundary.

For the two 5/8-weight operators, note that the identification of $\phi_{2,1}$ as a twist operator does not survive as we enter into the dense phase with $\kappa > 4$. In the dense phase each

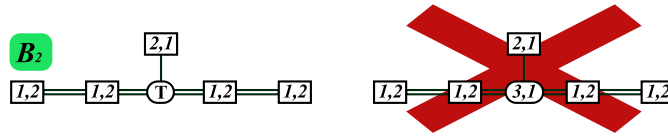


Figure 8. Using the same conventions as figure 7 we illustrate the potential conformal blocks of our percolation correlation function.

bond on the medial lattice is occupied and there is no difference in loop parity between different configurations, so $\phi_{2,1}$ belongs to the identity sector as the leading bulk energy density operator, a role it takes over from $\phi_{3,1}$ in the dilute phase. In the Q -state Potts model (of which percolation is the $Q \rightarrow 1$ limit) the energy density operator indicates there are two different species of spin locally. For two neighboring spins to be distinct the outer edges of their respective clusters must pass between them, implying that a total of four legs emanate from the energy density operator. At $Q = 1$ the weights of the loop configurations are independent of our spin labels and the energy density operator becomes equivalent to the 4-leg operator which was identified as $\phi_{0,2}$ in [17].

The configurations in the boundary percolation model described by the six-point function $\langle \phi_{1,2}(x_1)\phi_{1,2}(x_2)\phi_{1,2}(x_3)\phi_{1,2}(x_4)\phi_{2,1}(z, \bar{z}) \rangle$ are those with four SLE type paths flowing toward a single bulk point, which is a multiple radial SLE₆. As we bring the bulk 4-leg operator to the boundary we naturally expect its bulk–boundary fusion to have a boundary 4-leg operator leading term corresponding to the boundary operator with weight $h_{1,5} = 2$. It is the regular channel of $\phi_{2,1} \times \phi_{2,1} = \mathcal{I}_{3,1}$ that has the weight 2 highest weight operator T .

This is indeed the fusion channel of the B_2 block, which leads to the correlation function

$$\frac{\text{Im}(z)^{3/4} [(x_4 - x_3)(x_4 - x_2)(x_4 - x_1)(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)]^{1/3}}{|z - x_4||z - x_3||z - x_2||z - x_1|}. \tag{90}$$

We illustrate the single allowed conformal block B_2 in figure 8.

Again we see that the allowed block is formed when the bulk CFT fuses to the boundary CFT exclusively via the identity block. This rule leads to different unique solutions for the SLE₆ and SLE_{8/3} interpretation of our chiral correlation function due to the interchange of bulk and boundary sub-theories between the two models. In both cases observing this rule is sufficient to isolate the physical solution from the set of conformal blocks.

We end by suggesting that this construction may apply to a wider variety of extended minimal model BCFTs. It is perhaps natural to assume that two different logarithmic partners may reside in the bulk and on the boundary and that the consistency of the theory may be based on restricting the allowed bulk–boundary fusions. But then the allowed fusion products should include the entire minimal model Kac table, since we expect the extended CFT to include the minimal model as a sub-sector. However, this is essentially what we have done above, since the $c = 0$ minimal model is simply the identity.

6. Conclusions

In this paper we have assembled a variety of $O(n)$ model correlation functions containing twist operators. The correlation functions included correspond to: a loop anchored at two points with a pair of twist operators, an SLE with a pair of twist operators and a double SLE with a pair of twist operators. We then take these correlation functions in the $n \rightarrow 0$ limit where the $O(n)$ model corresponds to self-avoiding loops or dilute polymers. We verify that

in this limit the twist operator can play the role of zero weight indicator operator introduced by Schramm by confirming that the limiting forms of the SLE and twist operator correlation functions are equivalent to known results using the indicator operator.

We then use the extra constraints imposed by the twist operator to solve the correlation function of a new SLE type result with two identified bulk points. This result determines the winding distribution of an $SLE_{8/3}$ with respect to two marked boundary points.

In addition to determining a novel result for the self-avoiding walk, the calculation of the conformal blocks for the chiral 6-point function $\langle \phi_{2,1} \phi_{2,1} \phi_{2,1} \phi_{2,1} \phi_{1,2} \phi_{1,2} \rangle$ allows us to discuss a relevant problem in the logarithmic CFT with $c = 0$. The problem arises from the inability of both $\mathfrak{M}_{1,2}$ and $\mathfrak{M}_{2,1}$ to exist in the same chiral theory, in spite of the fact that both of these modules correspond to meaningful observables which can coexist in the self-avoiding loop and percolation models, both of which correspond to $c = 0$. In this paper we conjecture that the resolution to this apparent contradiction is that the two incompatible modules must belong to different sectors whether bulk or boundary, and that the bulk–boundary fusions must be through the identity channel common to both. This is consistent with the conformal blocks found in the text and their interpretations as physical quantities. It is our intention to further catalog the roles played by the logarithmic operators in both the CFTs of boundary percolation and the self-avoiding loop in a forthcoming paper.

We further conjecture that this coexistence of incompatible modules on the bulk and boundary may also apply to other extended minimal models such as the $O(1)$ model with $c = 1/2$ under the condition that the bulk–boundary fusions occur via the minimal model subsector.

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